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Inverse avalanches on Abelian sandpiles

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A simple and computationally efficient way of finding inverse avalanches for Abelian sandpiles, called the inverse particle addition operator, is presented. In addition, the method is shown to be optimal in the sense that it requires the minimum amount of computation among methods of the same kind. The method is also conceptually succinct because avalanche and inverse avalanche are placed in the same footing.

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The Abelian sandpile model (ASM), whose mathematical structure is first studied extensively by Dhar [1], is one of the few classes of models of self-organized criticality in which alot of interesting physical properties can be found analytically. The model consists of a finite number of sites labeled by an index set I. For each site $i \in I$, we assign an integer h_i, called the local height, to it. Whenever the local height of a site exceeds a threshold (which is fixed to 0 for simplicity), the site is called unstable and will transport some of its local heights (or sometimes called particles at that site) to the other sites in the next time step by

$$h_j \longrightarrow h_j - \Delta_{ij}, \qquad \text{whenever } h_i > 0. \tag{1}$$

Δ is called the toppling matrix whose elements satisfies

$$\Delta_{ii} > 0, \qquad \forall\, i \in I, \tag{2a}$$

$$\Delta_{ij} \leq 0, \qquad \forall\, i \neq j, \tag{2b}$$

and

$$\sum_{j\in I} \Delta_{ij} \geq 0, \qquad \forall\, i \in I. \tag{2c}$$

Toppling is repeated until all sites become stable again. The whole process of toppling is collectively known as an avalanche. The system is driven by adding a unit amount of particles onto the sites randomly and uniformly after the system regains its stability.

A system configuration, stable or not, can be regarded as a point in the space $\mathcal{Z}^N$ where N is the total number of sites in the system. Both the addition of a particle and the toppling of particles in a site can be regarded as a translation in $\mathcal{Z}^N$ [1–3]. The process of adding a particle to the site i together with the subsequent toppling it triggers (if any) can be viewed as a map between the set of all stable system configurations, and is denoted by $\mathbf{a}_i$ [1,4].

Based on the observation that the final stable state is independent of the order of toppling in different sites, Dhar shows that $\mathbf{a}_i \circ \mathbf{a}_j(\alpha) = \mathbf{a}_j \circ \mathbf{a}_i(\alpha)$ for any stable system configuration α [1]. This is why we call the model "Abelian." Using this commutative property, the total number of recurrence states in the model is shown to be $\det \Delta$ [1], and we denote the set of all recurrence states by Ω.

Generalization of the ASM, known as the generalized Abelian sandpile model (GASM), has been made recently by Chau and Cheng [2]. In their model, the local heights h_i and the elements in the toppling matrix Δ are real numbers instead of integers. Also, Δ satisfies only Eqs. (2a) and (2b). An arbitrary amount of particles are allowed to add to the system in possibly different locations all at the same time. Moreover, some special kind of configuration dependent triggering thresholds are used to determine the local stability of the pile (see Ref. [2] for details). In spite of the large differences between the ASM and the GASM, similar commutative properties between particle addition operators are found for recurrence system configurations.

In both the ASM and the GASM, one can prove that for any pair of $\alpha \in \Omega$ and $\mathbf{a}_i$, there exists a unique system configuration $\beta \in \Omega$ such that $\mathbf{a}_i(\beta) = \alpha$ [1,4]. While the avalanche problem (i.e., the problem of finding α given β and $\mathbf{a}_i$) is straightforward and can be done very quickly in computer; the inverse avalanche problem (i.e., the problem of finding β given α and $\mathbf{a}_i$) is much more difficult. β cannot be found, in general, by simply removing particle(s) from site(s) because the particle removal operation can map a state out of the eventual phase space Ω. Furthermore, the relationship $\mathbf{a}_i^{-1} = \mathbf{a}_i^{\det\Delta - 1}$ does not work very well for two reasons. First, $\det\Delta$ is, in general, a huge number making the method computationally impractical. Second, in the event of the GASM, $\det\Delta$ may not be an integer and hence $\mathbf{a}_i^{\det\Delta}$ is not well defined.

The first computationally feasible method of finding inverse avalanches is proposed recently by Dhar and Manna using the so-called inverse avalanche operator by means of the burning algorithm [5]. However, the method works only on the ASM with a symmetric toppling matrix. In this paper, we introduce a simple method to tackle the inverse avalanche problem for both the ASM and the GASM. To each particle addition operator, we find a succinct and simple way to associate its inverse to another particle addition operator, called the inverse particle addition operator. Then we prove that the method is computationally optimal in the sense that it requires the least number of topplings among all the possible methods using the idea of inverse particle addition operators.

For simplicity, we concentrate only on the case of Abelian sandpiles in the discussions below. However, all arguments, after slight modifications, work equally well on the GASM. We represent a system configuration α, stable or not, by a row vector of length N. In particular, the marginally stable state of the system (i.e., the one to which an avalanche is triggered whenever particles are added to any one of the sites) is $\vec{0} \equiv (0, 0, \ldots, 0)$.

Given a particle addition operator $\mathbf{a}$, we consider

$$\gamma \equiv (\gamma_j)_{j\in I} \equiv \mathbf{a}(\vec{0}) = \left(a_j - \textstyle\sum_k n_k \Delta_{kj}\right)_{j\in I} \in \Omega, \quad (3)$$

where $a_i \geq 0$ is the number of particles added to site i, and $n_i \in \mathcal{N}$ for all $i \in I$ is the total number of toppling in site i triggered after the particles are added [3]. For example, $a_j = \delta_{ij}$ for the operation of adding a single particle to site i together with the subsequent toppling induced $\mathbf{a}_i$. The above definition works equally well when more than one particle is introduced to the system each time, and when they are introduced to different sites.

Consider the operation of adding $-\gamma_i$ particles to site i for all $i \in I$ together with the subsequent toppling induced (if any). We denote this operation by $\tilde{\mathbf{a}}$. This is a well-defined operation sending system configurations from Ω to Ω because $\gamma_i \leq 0$ for all i. For any $\alpha \equiv (\alpha_i)_{i\in I} \in \Omega$,

$$\tilde{\mathbf{a}}(\alpha) = \left(\alpha_j - a_j + \textstyle\sum_k n_k{}' \Delta_{kj}\right)_{j\in I} \in \Omega \quad (4)$$

for some $n_k{}' \in \mathcal{Z}$. Moreover,

$$\mathbf{a} \circ \tilde{\mathbf{a}}(\alpha) = \left(\alpha_j + \textstyle\sum_k n_k{}'' \Delta_{kj}\right)_{j\in I} \in \Omega \quad (5)$$

for some $n_k{}'' \in \mathcal{Z}$. But by the remark between corollary 1 and corollary 2 in Ref. [2], we conclude that $n_k{}'' = 0$ for all $k \in I$ and hence $\mathbf{a} \circ \tilde{\mathbf{a}}(\alpha) = \alpha$ for all $\alpha \in \Omega$. Since the particle addition operators commute with each other, we can also conclude that $\tilde{\mathbf{a}} \circ \mathbf{a}(\alpha) = \alpha$. As a result, $\tilde{\mathbf{a}} \equiv \mathbf{a}^{-1}$ is the inverse particle addition operator corresponding to the particle addition operator $\mathbf{a}$.

We proceed to show that the above way of finding inverse avalanches is computationally optimal in the sense that the total number of toppling involved in the calculation is minimum among all the inverse particle addition operators (such as $\mathbf{a}^{\det\Delta - 1}$).

Suppose $\mathbf{b}'$ is another inverse particle addition operator corresponding to the particle addition operator $\mathbf{a}$ consisting of adding $b_i(\geq 0)$ particles to site i for all $i \in I$ together with the subsequent toppling. Clearly, $b_j = \sum_k m_k \Delta_{kj} - a_j \geq 0$ for some $m_k \in \mathcal{N}$ [2]. Consider the addition of b_i particles into site i' for all i at the same time to the system configuration γ. The resultant configuration is $\mu \equiv \left[\sum_k (m_k - n_k)\,\Delta_{kj}\right]_{j\in I}$. Since $\mathbf{b}'(\gamma) = \mathbf{a}^{-1}(\gamma) = \vec{0}$, μ must either equal to $\vec{0}$ or it is an unstable configuration which will eventually topple to $\vec{0}$. In either case, we can conclude that $m_i - n_i \geq 0$ for all $i \in I$ and the equality holds if and only if $b_i = -\gamma_i$ for all $i \in I$.

For any $\alpha \in \Omega$, the introduction of b_i particles to site i for all i is equivalent to the addition of first $-\gamma_i$ particles to site i for all i and then $\sum_k (m_k - n_k)\Delta_{ki}$ particles into site i for all i. So as compared to $\tilde{\mathbf{a}}$, $\mathbf{b}'$ requires $\sum_i (m_i - n_i)$ more toppling to find the inverse avalanche. Thus $\tilde{\mathbf{a}}$ is computationally optimal.

In summary, we have introduced a simple and efficient way to find the inverse avalanche for both the Abelian sandpile and the generalized Abelian sandpile by means of inverse particle addition operator. The method is conceptually succinct because inverse particle addition operators are placed in the same footing as the particle addition operators.

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